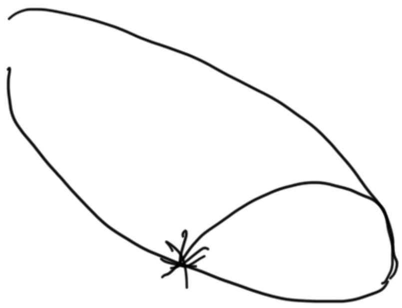
Intro to GR hydrodynamics



What is different when $v \sim c$?

Newtonian

$$\frac{d\rho}{dt} = -\rho(\nabla \cdot \vec{v})$$

$$\frac{d\vec{v}}{dt} = -\frac{\nabla P}{\rho} - \nabla\Phi$$

$$\frac{de}{dt} = -\frac{\nabla \cdot (P\vec{v})}{\rho}$$

$$\nabla^2\Phi = 4\pi G\rho$$

e.g. $\Phi_{BH} = -\frac{GM_{BH}}{r}$

$$\frac{d\rho}{dt} = \frac{\partial\rho}{\partial t} + \vec{v} \cdot \nabla\rho$$

GR

$$\frac{d\rho^*}{dt} = \rho^* \frac{\partial v^i}{\partial x^i}$$

$$\frac{dp_i}{dt} = -\frac{1}{\rho^*}\frac{\partial \sqrt{-g}P}{\partial x^i} + \frac{\sqrt{-g}}{2\rho^*} T^{\mu\nu} \frac{\partial g_{\mu\nu}}{\partial x^i}$$

$$\frac{de}{dt} = -\frac{1}{\rho^*}\frac{\partial}{\partial x^i}\left(\sqrt{-g}Pv^i\right) - \frac{\sqrt{-g}}{2\rho^*} T^{\mu\nu} \frac{\partial g_{\mu\nu}}{\partial t}$$

$$G_{\mu\nu} = 8\pi G T_{\mu\nu}$$

$\frac{d\vec{x}}{dt} = \vec{v}$

$c = 1$

The key aspects are:

i) We must define corresponding conserved mass, momentum and energy, which differ from Newtonian counterparts

$$\rho, \vec{v}, e \to (\rho^*, p_i, e)$$

ii) We require a procedure to solve for $\boxed{\rho, \vec{v}, e \text{ from } \rho^*, p_i, e}$

iii) Newtonian gravitational force $(-\nabla\Phi)$ replaced by gradients of metric $\boxed{\frac{\partial g_{\mu\nu}}{\partial x^i}}$

Recall that

$$ds^2 = g_{\mu\nu}\, dx^\mu\, dx^\nu$$

e.g. Minkowski metric

$$ds^2 = -c^2 dt^2 + dx^2 + dy^2 + dz^2$$

$$g_{\mu\nu} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & \end{pmatrix}$$

$$\begin{bmatrix} \cdots \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

"gravitational forces" are zero

iv) we adopt standard GR conventions e.g. Einstein summation convention

$$\frac{\partial v^i}{\partial x^i} \equiv \sum_{i=1}^{3} \frac{\partial v^i}{\partial x^i} = \frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} + \frac{\partial v_z}{\partial z}$$

v) In general we must solve Einstein equations to get metric

Recall

$$T^{\mu\nu} = (\rho c^2 + \rho u + P)\, U^\mu U^\nu + P g^{\mu\nu}$$

$$T_{\mu\nu} = (\rho c^2 + \rho u + P)\, \underline{U_\mu}\, \underline{U_\nu} + P g_{\mu\nu}$$

where $U^\mu \equiv \dfrac{dx^\mu}{d\tau}$ τ = proper time

we have $\boxed{v^\mu = \dfrac{dx^\mu}{dt}}$ $\boxed{U^0 = \dfrac{dt}{d\tau}}$

since $U^\mu = U^0 v^\mu$

...

hence $T_{\mu\nu}$ relates to our "primitive quantities" $\rho, \vec{v}, u, P$

For TDEs we can simplify matters by using exact solutions to Einstein's eq^s. For example in Newtonian gravity:

$$\Phi = -\frac{GM}{r}$$

GR equivalent is the Schwarzschild metric

$$\boxed{ds^2 = -\left(1 - \underbrace{\frac{2GM}{rc^2}}_{-\frac{2\Phi}{c^2}}\right)c^2dt^2 + \frac{dr^2}{\left(1-\frac{2GM}{rc^2}\right)} + r^2\left(d\theta^2 + \sin^2\theta d\phi^2\right)}$$

Recall also that $ds^2 = -c^2 d\tau^2$

In Cartesian coordinates, with $c=1$

$$g_{\mu\nu} = \begin{bmatrix} -f & 0 & 0 & 0 \\ 0 & f^{-1}\left[1 - \frac{2M}{r^3}(y^2+z^2)\right] & \frac{xy}{f}\frac{2M}{r^3} & \\ \dots \end{bmatrix}$$

$$\frac{xy}{\gamma}\frac{2m}{r^3} \quad \cdots \quad -(x^2+z^2)$$

Hence we can easily compute

$\boxed{\dfrac{\partial g_{\mu\nu}}{\partial x^i}}$ that we need

$$\gamma \equiv \left(1 - \frac{2GM}{rc^2}\right)$$

Also $\sqrt{-g} = 1$ in cartesians

$$\int \rho \, dV = \iiint \rho \, dx\,dy\,dz = \iiint \rho \sqrt{-g}\, dr\,d\theta\,d\phi$$

Conserved quantities in GR

e.g. mass conservation

$\int \rho \, dV$ in Newtonian physics

In GR

$\int \rho \sqrt{-g} \, d^3\vec{r}$ for general coordinate system

$$\rho^* = \rho\sqrt{-g}$$

Similarly if we consider special relativity

$$M = \int \rho \gamma \, dV \qquad \gamma = \frac{1}{\sqrt{1-\frac{v^2}{c^2}}} = \frac{dt}{d\tau}$$

In gr we generalise to

$$M = \int \rho U^0 \sqrt{-g}\, d^3x$$

Hence we define

$$\boxed{\rho^* \equiv \rho \underline{U}^0 \sqrt{-g}}$$

$$-c^2 d\tau^2 = g_{\mu\nu}\, dx^\mu dx^\nu$$

$$\therefore \left(\frac{d\tau}{dt}\right)^2 = -g_{\mu\nu} \frac{dx^\mu}{dt} \frac{dx^\nu}{dt}$$

$$= -g_{\mu\nu} \underline{v}^\mu \underline{v}^\nu$$

Similarly:

$$\boxed{p_i = U^0 w\, g_{i\mu} v^\mu}$$

$$w \equiv 1 + u + \frac{p}{\rho}$$

$$\boxed{e = v^i p_i + \frac{1+u}{U^0}}$$

Can derive conserved quantities by starting with Lagrangian

$$L_m = \int \underline{T^{\mu\nu} U_\mu U_\nu \, dV}$$

$$\boxed{\frac{d}{dt}\left(\frac{\partial L}{\partial v^i}\right) = \frac{\partial L}{\partial x^i}}$$ Euler-Lagrange equations

e.g. $\boxed{p_i \equiv \frac{\partial L}{\partial v^i}}$ canonical momentum

<u>Tidal disruption events</u>

$$R_t = R_* \left(\frac{M_{BH}}{M_*} \right)^{1/3}$$

$$\frac{M_{BH}}{R_{BH}^3} = \frac{M_*}{R_*^3}$$

Outside R_t, self-gravity of star is important $\Rightarrow$ how to do this?

Write metric in form

$$g_{\mu\nu} = \bar{g}_{\mu\nu} + h_{\mu\nu}$$

$$|h_{\mu\nu}| \ll \bar{g}_{\mu\nu}$$

$\bar{g}_{\mu\nu}$ = Kerr or Scharzschild

Consider that in limit of Newtonian gravity, we would have

$$ds^2 = -\underline{\left(1 + \frac{2\Phi}{c^2}\right)} c^2 dt^2 + \cancel{\left(1 - \frac{2\Phi}{c^2}\right)} dx^2 + dy^2 + d$$

$$h_{\mu\nu} = \begin{bmatrix} \frac{2\Phi}{c^2} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Since $|h_{\mu\nu}| << \bar{g}_{\mu\nu}$ can neglect $h_{\mu\nu}$ everywhere except for derivatives

$$\frac{dP_i}{dt} = -\frac{1}{\rho^*}\frac{\partial}{\partial x^i}\left(\sqrt{-g}P\right) + \frac{\sqrt{-g}}{2\rho^*}T^{\mu\nu}\left[\frac{\partial \bar{g}_{\mu\nu}}{\partial x^i} + \frac{\partial h_{\mu\nu}}{\partial x^i}\right.$$

If we expand out, get

$$\left.\frac{dP_i}{dt}\right|_{sg} = \frac{1}{2}u^0 v^\mu v^\nu \frac{\partial h_{\mu\nu}}{\partial x^i} \qquad v^0 \equiv \frac{dt}{dt} =$$

$$= \frac{1}{2}u^0 \frac{\partial h_{00}}{\partial x^i}$$

$$= -u^0 \frac{\partial \Phi}{\partial x^i}$$

$$\simeq -\nabla\Phi$$

Now we can simulate tidal

disruption events!!

e.g. in SPH we have a set of points $\boxed{\frac{dx^i}{dt} = v^i}$